Journal of Engineering Mathematics 18 (1984) 315–333.

A diffusion problem in semiconductor technology

J. BOERSMA *, J.J.E. INDENKLEEF * and H.K. KUIKEN **

* *Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, The Netherlands*
** *Philips Research Laboratories, Eindhoven, The Netherlands*

(Received June 4, 1984)

Summary

This paper deals with a mathematical model of a SEM-EBIC experiment devised to measure the diffusion length of semiconductor materials. In the model the semiconductor material occupies a half-space, of which the plane bounding surface is partly covered by a semi-infinite current-collecting junction, the Schottky diode. A scanning electron microscope (SEM) is used to inject minority carriers into the material. It is assumed that injection occurs at a single point only. The injected minority carriers diffuse through the material and recombine in the bulk at a rate proportional to their local concentration. Recombination also occurs at the free surface of the material, not covered by the junction, where its rate is determined by the surface recombination velocity v. The mathematical model gives rise to a mixed-boundary-value problem for the diffusion equation, which is solved by means of the Wiener-Hopf technique. An analytical expression is derived for the measurable electron-beam-induced current (EBIC) caused by the minority carriers reaching the junction. The solution obtained is valid for all values of v, and special attention is given to the limiting cases $v = \infty$ and $v = 0$. Asymptotic expansions of the induced current, which are usable when the injection point is more than a few diffusion lengths away from the edge of the junction, are derived as well.

1. Introduction

The electrical properties of semiconductor materials are determined to a large extent by the transport properties of the minority carriers [1]. If the doping of the material is such that it is of n-type, then the holes, which can be considered as positively charged, are the minority carriers. In p-type material this role is played by the electrons. If minority carriers are injected into semiconductor material, there is a tendency for them to diffuse away from the point of injection. This process is characterized by the diffusion coefficient D. On the other hand, being immersed in a region where charges of the opposite kind abound, the minority carriers show a propensity towards annihilation by way of a process called recombination. The latter process is characterized by the lifetime τ in the sense that the recombination occurs more rapidly as τ gets smaller, Let $C = C(\boldsymbol{x}, t)$ denote the concentration of the minority carriers at position $\boldsymbol{x} \in \mathbb{R}^3$ or $\mathbb{R}^2$ and time t. Then the combined processes of diffusion and recombination are governed by the diffusion equation

$$\frac{\partial C}{\partial t} = D\Delta C - \frac{C}{\tau} + Q(\boldsymbol{x}, t), \tag{1.1}$$

in which the source term Q describes the generation rate of minority carriers per unit volume. In the steady state Eqn. (1.1) reduces to

$$D\Delta C - \frac{C}{\tau} = -Q(x) \tag{1.2}$$

where C and Q are independent of the time t.

On the basis of the two parameters D and τ one may define a characteristic length

$$L = (D\tau)^{1/2} \tag{1.3}$$

which is called the diffusion length. This parameter is of particular interest to semiconductor technologists as it is a measure of the distance over which the concentration drops an order of magnitude. To determine the parameter L, simple experiments have been devised involving configurations for which a closed-form solution of Eqn. (1.2) is available. The most widely used configuration is the one shown in Fig. 1. Here, a diode consisting of both p- and n-type material is used. As its dimensions exceed the diffusion length by at least one order of magnitude, the diode may be considered semi-infinite. In terms of Cartesian coordinates x, y, z, the diode occupies the half-space $y \geqslant 0$ and its junction is located at $x = 0$. By means of a scanning electron microscope (SEM) minority carriers, which are holes in the case of Fig. 1, are injected at the point (x_0, y_0, z_0). The holes reaching the junction cause an electron-beam-induced current (EBIC)

$$I = \int_{-\infty}^{\infty} \mathrm{d}z \int_0^{\infty} D \frac{\partial C}{\partial x}(0, y, z) \mathrm{d}y \tag{1.4}$$

which can be measured. The variation of the current I caused by varying the point of injection gives information that may lead to the determination of the diffusion length L.

The mathematical problem corresponding to the configuration of Fig. 1 requires the solution of Eqn. (1.2) subject to the boundary conditions

$$C(0, y, z) = 0, \quad D\frac{\partial C}{\partial y}(x, 0, z) = vC(x, 0, z) \tag{1.5}$$

at the junction and the free surface, respectively. The parameter v is the so-called surface

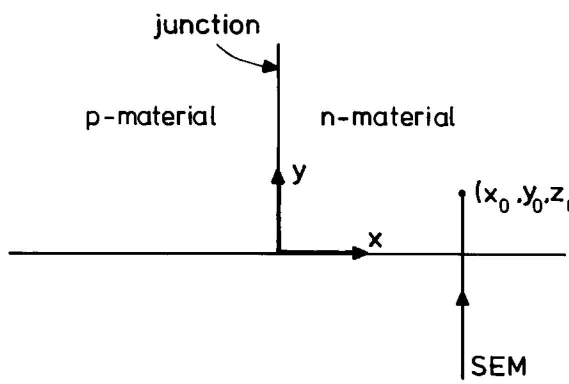

Figure 1. Configuration for diffusion length measurement by means of a p-n junction.

recombination velocity, and the boundary condition at the free surface expresses that the surface is a sink for the minority carriers. This is due to a recombination phenomenon over and above that prevailing in the bulk,and existing in an extremely thin surface layer. The solution of problem (1.2), (1.5), is readily obtained by the method of images, as already recognized in the early days of semiconductor technology by, for example, Van Roosbroeck [2] who derived the Green's function pertinent to the problem (1.2), (1.5). Since then the matter remained dormant for about two decades before interest in this problem area was renewed. As a result numerous papers, [3,4,5,6,7] to name a few, have been published, while a recent survey may be found in [8]. In retrospect it is clear that the successful use of the configuration of Fig. 1 as an experimental device is to be attributed mainly to the fact that a closed-form analytical solution of the corresponding mathematical problem is available. Thus a careful experimentation using this device can lead to meaningful results.

In this paper we consider a different configuration, depicted in Fig. 2, which is often more suitable from an experimental point of view. In this case pure semiconductor material, either of p- or n-type, is used. In terms of Cartesian coordinates x, y, z, the material fills the half-space $y \geqslant 0$. The plane boundary $y = 0$ is partly covered by a semi-infinite Schottky diode located at $x \leqslant 0$, $y = 0$. Inside the material minority carriers are injected at the point (x_0, y_0, z_0) by a SEM electron beam as before. The resulting current collected at the Schottky diode is given by

$$I = \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{0} D \frac{\partial C}{\partial y}(x, 0, z) \mathrm{d}x \tag{1.6}$$

which can again be measured.

The present configuration has received much less attention than that of Fig. 1, probably due to the fact that no analytical solution of the corresponding mathematical problem has been available hitherto. It is the aim of this paper to present such a solution. The mathematical problem requires again the solution of Eqn. (1.2), but now subject to the boundary conditions

$$C(x, 0, z) = 0, \quad x < 0; \quad D \frac{\partial C}{\partial y}(x, 0, z) = vC(x, 0, z), \quad x > 0, \tag{1.7}$$

at the Schottky diode and the remaining free surface, respectively; here, v denotes the surface recombination velocity as before. The problem described by Eqns. (1.2), (1.7), is a mixed boundary value problem which in general cannot be solved by elementary means. This may explain why the analytical solution of the problem has not yet been derived. The

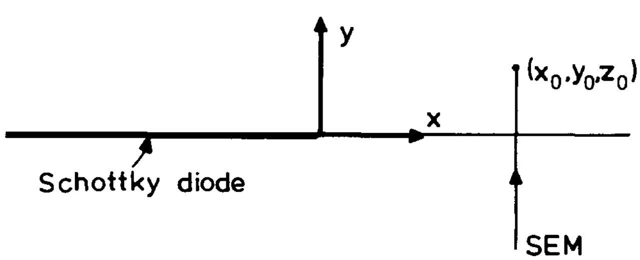

Figure 2. Configuration for diffusion length measurement by means of a Schottky diode.

only inroad that has been made so far concerns the special case $v = \infty$. Then the second boundary condition in (1.7) reduces to $C = 0$ and this condition is seen to prevail along the entire boundary $y = 0$. The method of images can now be brought into play and a simple solution is readily obtained [9,10,11].

In this paper we shall present the solution to the complete problem, i.e. with arbitrary constant v in the range from 0 up to ∞. The source term $Q(\boldsymbol{x})$ in (1.2) is taken as either a line source at (x_0, y_0), or a point source at (x_0, y_0, z_0), with $y_0 > 0$. It will turn out that the results for the current I are the same when the line and point sources are of equal strength. In Section 2 the problem is reformulated in a dimensionless form. It is pointed out that our problem resembles certain wave propagation and diffraction problems [12–16] which have been solved by Fourier transforms and the Wiener-Hopf technique [17]. The latter method is used in Section 3 to solve our diffusion problem. In Section 4 we derive an analytical expression for the current I. The expression obtained is in terms of a rather complicated integral, but its numerical evaluation presents no difficulty in principle. Section 5 deals with the limiting cases $v = \infty$ and $v = 0$, in which the previous results simplify considerably. Finally, in Section 6 we discuss the asymptotics of the current I if the distance from the point of injection to the edge of the Schottky diode is large.

Reference is made to the 'physical' companion paper [18] which has extensive numerical results for the current I. The main purpose of that paper is to present methods by which the diffusion length L and the dimensionless surface recombination velocity $\lambda = vLD^{-1}$ can both be determined from plots of the measured current versus the position of the injection point. Among the further topics treated in [18] we mention: the solution of the diffusion problem for a source of finite non-zero extent, the physical implications of the mathematical results and the use of the configuration of Fig. 2 as an experimental device. Summarizing, we feel that the two papers, [18] and this one, provide a nice example to show that the practicability of certain experiments strongly depends upon the availability of analytical solutions of mathematical models relevant to those experiments.

2. Statement of the problem

Consider the diffusion problem described by Eqns. (1.2), (1.7), in which the source term is taken either as a line source at (x_0, y_0), given by

$$Q(\boldsymbol{x}) = Q\delta(x - x_0)\delta(y - y_0), \quad y_0 > 0, \tag{2.1a}$$

or as a point source at (x_0, y_0, z_0), given by

$$Q(\boldsymbol{x}) = Q\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \quad y_0 > 0. \tag{2.1b}$$

The constant Q in (2.1b) measures the strength of the source, i.e. carrier production per unit time, whereas in (2.1a) Q stands for the source strength per unit length. Clearly, in the case (2.1a) of a line source there is no dependence on the coordinate z and the problem becomes two-dimensional. We now introduce the dimensionless coordinates and parameter

$$x' = L^{-1}x, \quad y' = L^{-1}y, \quad z' = L^{-1}z; \quad \lambda = vLD^{-1}, \tag{2.2}$$

and the dimensionless concentrations

$$p(x', y') = Q^{-1}DC(x, y), \quad p(x', y', z') = Q^{-1}LDC(x, y, z), \tag{2.3}$$

corresponding to the cases (2.1a) and (2.1b), respectively; here, L is the diffusion length defined in (1.3), and D is the diffusion constant. For simplicity we henceforth suppress the primes in the coordinates x', y', z'. Then the two-dimensional diffusion problem with injection by a line source can be reduced to the dimensionless form

$$\Delta p - p = -\delta(x - x_0)\delta(y - y_0), \quad -\infty < x < \infty, y > 0; \tag{2.4a}$$

$$p(x, 0) = 0, \quad x < 0; \tag{2.4b}$$

$$\frac{\partial p}{\partial y}(x, 0) - \lambda p(x, 0) = 0, \quad x > 0. \tag{2.4c}$$

Furthermore, to ensure uniqueness of the solution, it is required that $p \to 0$ as $x^2 + y^2 \to \infty$, or more specifically

$$p(x, y) = O(r^{-1/2} \mathrm{e}^{-r}), \quad r = (x^2 + y^2)^{1/2} \to \infty. \tag{2.4d}$$

The current through the Schottky diode is found to be given by the dimensionless expression

$$\frac{I}{Q} = \int_{-\infty}^{0} \frac{\partial p}{\partial y}(x, 0)\mathrm{d}x. \tag{2.5}$$

For the three-dimensional diffusion problem with injection by a point source, the dimensionless concentration $p(x, y, z)$ must satisfy the equation

$$\Delta p - p = -\delta(x - x_0)\delta(y - y_0)\delta(z - z_0), \quad -\infty < x < \infty, \quad y > 0, \quad -\infty < z < \infty, \tag{2.6}$$

and boundary and decay conditions similar to (2.4b, c, d). The dimensionless current is now given by

$$\frac{I}{Q} = \int_{-\infty}^{\infty} \mathrm{d}z \int_{-\infty}^{0} \frac{\partial p}{\partial y}(x, 0, z)\mathrm{d}x. \tag{2.7}$$

By comparing the two- and three-dimensional problems, it is recognized that their solutions are simply related by

$$\int_{-\infty}^{\infty} p(x, y, z)\mathrm{d}z = p(x, y). \tag{2.8}$$

Hence, the currents I/Q in (2.5) and (2.7) are the same for point and line source injection, as observed before by Van Roosbroeck [2], Van Opdorp [6]. Thus it is sufficient to solve the two-dimensional diffusion problem (2.4).

The mixed boundary value problem (2.4) resembles certain wave propagation and diffraction problems treated in the literature. In his study of water waves over a channel of infinite depth with a dock, Heins [12] was led to a problem identical to (2.4) without the source term in (2.4a), and with (2.4b) replaced by the condition that the normal derivative should vanish. Bazer and Karp [13] treated the propagation of a plane electromagnetic wave over a planar land-sea surface which is modelled by boundary conditions similar to (2.4b,c). Senior [14,15,16] studied the diffraction of a plane electromagnetic wave by an imperfectly conducting half-plane which is modelled by an impedance boundary condition of the form (2.4c). Notice that these diffraction problems are governed by the Helmholtz equation $\Delta u + k^2 u = 0$, hence, to achieve agreement with Eqn. (2.4a), the parameter k should be taken imaginary. Also the right analog of our diffusion problem with injection by a line source would be the diffraction problem for an incident cylindrical wave rather than a plane wave. However, this defect is not very essential, since a cylindrical wave may be expressed as a superposition of plane waves. All papers mentioned above have in common that the solution of the mixed-boundary-value problems is derived by means of Fourier transforms and the Wiener-Hopf technique [17]. Thus it is most obvious to use the same tools in the solution of the diffusion problem (2.4).

3. Solution by the Wiener-Hopf technique

As a first step in the solution of the diffusion problem (2.4) we eliminate the source term in (2.4a) by subtracting a suitable particular solution. Thus we set

$$p(x, y) = \frac{1}{2\pi} K_0\Big(\big((x - x_0)^2 + (y - y_0)^2\big)^{1/2}\Big) - \frac{1}{2\pi} K_0\Big(\big((x - x_0)^2 + (y + y_0)^2\big)^{1/2}\Big) + u(x, y), \tag{3.1}$$

in which K_0 denotes the modified Bessel function of the second kind of order zero. It is well known that $(2\pi)^{-1} K_0(\cdot)$ is a fundamental solution of Eqn. (2.4a). Moreover, the combination of two K_0 functions has been deliberately chosen to satisfy the boundary condition (2.4b). By substitution of (3.1) into (2.4) we arrive at the following boundary value problem for the function $u(x, y)$:

$$\Delta u - u = 0, \quad -\infty < x < \infty, \quad y > 0; \tag{3.2a}$$

$$u(x, 0) = \begin{cases} 0, & x < 0, \\ f(x), & x > 0; \end{cases} \tag{3.2b}$$

$$\frac{\partial u}{\partial y}(x, 0) - \lambda u(x, 0) = \begin{cases} g(x), & x < 0, \\ -\dfrac{y_0}{\pi} \dfrac{K_1\Big(\big((x - x_0)^2 + y_0^2\big)^{1/2}\Big)}{\big((x - x_0)^2 + y_0^2\big)^{1/2}}, & x > 0; \end{cases} \tag{3.2c}$$

$$u(x, y) = O(r^{-1/2} e^{-r}), \quad r = (x^2 + y^2)^{1/2} \to \infty. \tag{3.2d}$$

In (3.2b, c) the boundary conditions at $y=0$ have been extended to the full range $-\infty < x < \infty$ by the introduction of unknown auxiliary functions $f(x)$ and $g(x)$. Near the edge $x=0$ of the Schottky diode we require that

$$f(x)=O(1), \quad g(-x)=O(x^{-1/2}), \quad x \downarrow 0, \tag{3.3}$$

similar to the well-known edge condition from diffraction theory. The validity of the conditions (3.3) can be justified a posteriori.

To solve the problem (3.2) we employ Fourier transformation with respect to the variable x. Thus we define

$$U(w, y)=\int_{-\infty}^{\infty} u(x, y)\, \mathrm{e}^{\mathrm{i}wx}\mathrm{d}x, \tag{3.4}$$

then $U(w, y)$ is an analytic function of the complex variable w in the strip $-1 < \operatorname{Im} w < 1$, by (3.2d). Furthermore, we introduce the Fourier transforms

$$F_+(w)=\int_0^{\infty} f(x)\, \mathrm{e}^{\mathrm{i}wx}\mathrm{d}x, \quad G_-(w)=\int_{-\infty}^{0} g(x)\, \mathrm{e}^{\mathrm{i}wx}\mathrm{d}x, \tag{3.5}$$

where the subscripts + and − indicate that these functions are analytic in the upper half-plane $\operatorname{Im} w > -1$ and the lower half-plane $\operatorname{Im} w < 1$, respectively. This convention will be adopted throughout the paper. The Fourier transform of the Bessel function in (3.2c) is known from [19, form. 1.13(44)], viz.

$$D(w)=\frac{y_0}{\pi}\int_{-\infty}^{\infty} \frac{K_1\left(\left((x-x_0)^2+y_0^2\right)^{1/2}\right)}{\left((x-x_0)^2+y_0^2\right)^{1/2}}\, \mathrm{e}^{\mathrm{i}wx}\mathrm{d}x=\exp\left[\mathrm{i}x_0 w - y_0(w^2+1)^{1/2}\right]. \tag{3.6}$$

The corresponding Fourier integrals over the semi-infinite intervals $(-\infty, 0]$ and $[0, \infty)$ are denoted by $D_-(w)$ and $D_+(w)$, respectively, then $D(w)=D_-(w)+D_+(w)$. By using these results in (3.2) we are led to the Fourier transformed problem

$$\frac{\partial^2 U}{\partial y^2}-(w^2+1)U=0, \quad y>0; \tag{3.7a}$$

$$U(w, 0)=F_+(w); \tag{3.7b}$$

$$\frac{\partial U}{\partial y}(w, 0)-\lambda U(w, 0)=G_-(w)-D_+(w)=G_-(w)+D_-(w)-D(w). \tag{3.7c}$$

Since the function $U(w, y)$ must vanish as $y \to \infty$, the solution of Eqn. (3.7a) is easily seen to be

$$U(w, y)=A(w)\exp\left[-y(w^2+1)^{1/2}\right], \tag{3.8}$$

in which the function $A(w)$ is yet to be determined. The square root $(w^2+1)^{1/2}=(1-\mathrm{i}w)^{1/2}(1+\mathrm{i}w)^{1/2}$ stands for the principal value, specified by $-\pi < \arg(1 \mp \mathrm{i}w) < \pi$ in the complex w-plane with branch cuts along the imaginary axis from $-i\infty$ to $-i$ and from i to $i\infty$. In the cut w-plane one has $\operatorname{Re}(w^2+1)^{1/2}>0$, so that (3.8) is indeed the appropriate solution. By imposing the boundary conditions (3.7b, c) to the solution (3.8) we find

$$A(w)=F_+(w); \quad -\left[\lambda+(w^2+1)^{1/2}\right]A(w)=G_-(w)+D_-(w)-D(w). \tag{3.9}$$

Next, by elimination of $A(w)$ we obtain the functional equation

$$\left[\lambda+(w^2+1)^{1/2}\right]F_+(w)+G_-(w)+D_-(w)-D(w)=0, \tag{3.10}$$

valid in the strip $-1<\operatorname{Im} w<1$, which is amenable to solution by the Wiener-Hopf technique.

The fundamental step in the Wiener-Hopf procedure for solving (3.10) is the factorization of $\lambda+(w^2+1)^{1/2}$ through

$$(w^2+1)^{1/2}=(1-iw)^{1/2}(1+iw)^{1/2}, \tag{3.11}$$

$$H(w)=1+\lambda(w^2+1)^{-1/2}=H_+(w)H_-(w), \tag{3.12}$$

where it is required that $H_+(w)$ is analytic and non-zero in the upper half-plane $\operatorname{Im} w>-1$, while $H_-(w)$ is analytic and non-zero in the lower half-plane $\operatorname{Im} w<1$. the factors $H_\pm(w)$ can be determined by means of Noble's procedure [17, p. 15, Thm. C], yielding

$$H_+(w)=\exp\left[\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{\log\left[1+\lambda(z^2+1)^{-1/2}\right]}{z-w}\,dz\right],$$

$$\operatorname{Im} w>a, \quad -1<a<1, \tag{3.13}$$

and $H_-(w)=H_+(-w)$. It is remarked that essentially the same factorization as (3.12) is encountered in [12,13,14,15,16]. In these papers an alternative representation for the factors is derived which for our problem becomes

$$H_+(i\sin\alpha)=\left(\frac{\cos\alpha+\cos\chi}{1+\sin\alpha}\right)^{1/2}\exp\left[\frac{1}{2\pi}\int_{\chi-\alpha}^{\chi+\alpha}\frac{t\,dt}{\sin t}\right],$$

$$-\pi/2<\operatorname{Re}\alpha<\pi/2, \tag{3.14}$$

where $\chi=\arccos\lambda$. The present expression is closely related to the function f employed by Senior [15,16], and to the function Ψ_π introduced by Maliuzhinets [20]. According to Bazer and Karp [13] a representation of the form (3.14) was already given by V.A. Fock as early as 1944. In this paper we employ the representation (3.13) of $H_+(w)$, which appears to be more convenient for numerical purposes. Returning to the functional equation (3.10), we substitute (3.11), (3.12), and divide by $(1+iw)^{1/2}H_-(w)$, then we obtain

$$(1-iw)^{1/2}H_+(w)F_+(w)+\frac{G_-(w)+D_-(w)}{(1+iw)^{1/2}H_-(w)}-\frac{D(w)}{(1+iw)^{1/2}H_-(w)}=0. \tag{3.15}$$

As the next step we carry out the decomposition

$$\frac{D(w)}{(1+\mathrm{i}w)^{1/2}H_-(w)}=E_+(w)+E_-(w) \tag{3.16}$$

where $E_+(w)$ and $E_-(w)$ are analytic in the half-planes $\operatorname{Im} w>-1$ and $\operatorname{Im} w<1$, respectively. The components $E_\pm(w)$ can be determined by means of Noble's procedure [17, p. 13, Thm. B], yielding

$$E_+(w)=\frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}b}^{\infty+\mathrm{i}b}\frac{D(z)}{(1+\mathrm{i}z)^{1/2}H_-(z)}\frac{\mathrm{d}z}{z-w},\quad \operatorname{Im} w>b,\quad -1<b<1, \tag{3.17}$$

$$E_-(w)=-\frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c}\frac{D(z)}{(1+\mathrm{i}z)^{1/2}H_-(z)}\frac{\mathrm{d}z}{z-w},\quad \operatorname{Im} w<c,\quad -1<c<1. \tag{3.18}$$

Inserting (3.16) into (3.15), we rearrange the functional equation as

$$(1-\mathrm{i}w)^{1/2}H_+(w)F_+(w)-E_+(w)=-\frac{G_-(w)+D_-(w)}{(1+\mathrm{i}w)^{1/2}H_-(w)}+E_-(w), \tag{3.19}$$

valid in the strip $-1<\operatorname{Im} w<1$. We now invoke the standard Wiener-Hopf argument. The left- and right-hand sides of (3.19) are analytic functions in the half-planes $\operatorname{Im} w>-1$ and $\operatorname{Im} w<1$, respectively. These functions coincide in the common strip $-1<\operatorname{Im} w<1$. Therefore the two sides of Eqn. (3.19) define an entire function. From (3.3), (3.13), (3.17) and (3.18) we infer the asymptotic behaviour

$$F_+(w)=O(w^{-1}),\quad G_-(w)=O(w^{-1/2}),\quad D_-(w)=O(w^{-1}),$$
$$H_\pm(w)=O(1),\quad E_\pm(w)=O(w^{-1/2}) \tag{3.20}$$

as $w\to\infty$ in the respective region of analyticity. Thus both sides of (3.19) tend to 0 as $w\to\infty$ in the respective half-planes. Then by Liouville's theorem the entire function defined by (3.19) vanishes identically. This leads to the following solutions for F_+ and G_-:

$$F_+(w)=\frac{E_+(w)}{(1-\mathrm{i}w)^{1/2}H_+(w)};\quad G_-(w)=(1+\mathrm{i}w)^{1/2}H_-(w)E_-(w)-D_-(w). \tag{3.21}$$

Finally, from (3.9) and (3.21) we determine the function $A(w)$ in (3.8). Then by inverse Fourier transformation of $U(w,y)$ we obtain

$$u(x,y)=\frac{1}{2\pi}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c}\frac{E_+(w)}{(1-\mathrm{i}w)^{1/2}H_+(w)}\exp\left[-\mathrm{i}xw-y(w^2+1)^{1/2}\right]\mathrm{d}w,$$
$$-1<c<1, \tag{3.22}$$

as the solution of the problem (3.2). Combined with (3.1) this result completes the solution of the diffusion problem (2.4).

4. Current through the Schottky diode

In the present investigation the main interest is in the evaluation of the current through the Schottky diode, due to the injection of minority carriers by a line source at (x_0, y_0) or by a point source at (x_0, y_0, z_0). As pointed out in Section 2, both cases give rise to the same dimensionless current I/Q given by (2.5) and (2.7). Using the boundary conditions (2.4b, c) we rewrite (2.5) as

$$\frac{I}{Q}=\int_{-\infty}^{\infty}\left[\frac{\partial p}{\partial y}(x,0)-\lambda p(x,0)\right]\mathrm{d}x. \tag{4.1}$$

Replace p by (3.1), then the resulting integral is recognized as the sum of the Fourier transforms (3.6) and (3.7c) with $w=0$, so that

$$\frac{I}{Q}=G_-(0)+D_-(0). \tag{4.2}$$

Substitute G_- from (3.21), then we obtain

$$\frac{I}{Q}=H_-(0)E_-(0)=(1+\lambda)^{1/2}E_-(0), \tag{4.3}$$

or, by means of (3.6) and (3.18),

$$\frac{I}{Q}=-\frac{(1+\lambda)^{1/2}}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c}\frac{\exp\left[\mathrm{i}x_0z-y_0(z^2+1)^{1/2}\right]}{(1+\mathrm{i}z)^{1/2}H_-(z)}\frac{\mathrm{d}z}{z},\quad 0<c<1. \tag{4.4}$$

This result will now be simplified in the two cases $x_0 \geqslant 0$ and $x_0 \leqslant 0$.

If $x_0 \geqslant 0$, the integration path in (4.4) is deformed into a loop around the branch cut from i to i∞. Beforehand, we replace the denominator $(1+\mathrm{i}z)^{1/2}H_-(z)$ by $[\lambda+(z^2+1)^{1/2}]/[(1-\mathrm{i}z)^{1/2}H_+(z)]$. By setting $z=\mathrm{i}t$ along the branch cut, we arrive at the representation

$$\frac{I}{Q}=\frac{(1+\lambda)^{1/2}}{\pi}\int_1^{\infty}(1+t)^{1/2}\exp[F(\lambda,t)]\,\mathrm{e}^{-x_0t}$$
$$\times\frac{\lambda\sin\left(y_0(t^2-1)^{1/2}\right)+(t^2-1)^{1/2}\cos\left(y_0(t^2-1)^{1/2}\right)}{\lambda^2+t^2-1}\frac{\mathrm{d}t}{t}, \tag{4.5}$$

valid for $x_0 \geqslant 0$. Here, the function $F(\lambda, t)$ is a short notation for $\log H_+(\mathrm{i}t)$. Hence, by means of (3.13) with $a=0$ and applying the substitution $(z^2+1)^{-1/2}=\sin\varphi$, we have

$$F(\lambda,t)=\frac{t}{\pi}\int_0^{\infty}\frac{\log\left[1+\lambda(z^2+1)^{-1/2}\right]}{z^2+t^2}\mathrm{d}z=\frac{t}{\pi}\int_0^{\pi/2}\frac{\log(1+\lambda\sin\varphi)}{\cos^2\varphi+t^2\sin^2\varphi}\mathrm{d}\varphi. \tag{4.6}$$

If $x_0 \leqslant 0$, the integration path in (4.4) is deformed into a loop around the branch cut from $-\mathrm{i}\infty$ to $-\mathrm{i}$. In the deformation a simple pole at $z=0$ is intercepted and its residue contribution should be accounted for. By setting $z=-\mathrm{i}t$ along the branch cut and remembering that $H_-(-\mathrm{i}t)=H_+(\mathrm{i}t)$, we obtain the representation

$$\frac{I}{Q}=\mathrm{e}^{-y_0}-\frac{(1+\lambda)^{1/2}}{\pi}\int_1^{\infty}(1+t)^{-1/2}\exp[-F(\lambda,t)]\,\mathrm{e}^{x_0 t}\sin\left(y_0(t^2-1)^{1/2}\right)\frac{\mathrm{d}t}{t},\tag{4.7}$$

valid for $x_0\leqslant 0$, in which $F(\lambda, t)$ is given by (4.6). Although rather complicated, the expressions (4.5) and (4.7) are well-suited for a further evaluation by numerical integration. Numerical results for the current I/Q are reported in [18].

5. Limiting cases $\lambda=\infty$ and $\lambda=0$

In the limiting cases $\lambda=\infty$ and $\lambda=0$ the previous results can be considerably simplified. The case $\lambda=\infty$ corresponding to an infinite surface recombination velocity v is of particular interest, since it is the only case in which the diffusion problem (2.4) has been solved in the literature [9,10,11]. When $\lambda=\infty$, the boundary conditions (2.4b, c) reduce to $p(x, 0)=0$ applied along the entire boundary $y=0$. Then the method of images supplies the solution in a simple manner, viz.

$$p(x,y)=\frac{1}{2\pi}K_0\left(\left((x-x_0)^2+(y-y_0)^2\right)^{1/2}\right)-\frac{1}{2\pi}K_0\left(\left((x-x_0)^2+(y+y_0)^2\right)^{1/2}\right).\tag{5.1}$$

From (2.5) the dimensionless current is found to be

$$\frac{I}{Q}=\frac{y_0}{\pi}\int_{x_0}^{\infty}\frac{K_1\left(\left(x^2+y_0^2\right)^{1/2}\right)}{\left(x^2+y_0^2\right)^{1/2}}\mathrm{d}x\tag{5.2}$$

which is identical to the expression derived by Ioannou and Dimitriadis [11]. The same result can also be obtained from (4.5) and (4.7) by taking the limit as $\lambda\to\infty$. To that end we first establish the auxiliary result

$$\begin{aligned}F(\lambda,t)&=\frac{t}{\pi}\int_0^{\pi/2}\frac{\log\lambda+\log\sin\varphi}{\cos^2\varphi+t^2\sin^2\varphi}\mathrm{d}\varphi+o(1)\\&=\tfrac{1}{2}\log\lambda-\tfrac{1}{2}\log(1+t)+o(1),\quad \lambda\to\infty\end{aligned}\tag{5.3}$$

where the elementary integrals have been quoted from [21, form 3.647, 4.385(3)]. Using (5.3) in (4.5) and (4.7), we find in the limiting case $\lambda=\infty$,

$$\frac{I}{Q}=\frac{1}{\pi}\int_1^{\infty}\mathrm{e}^{-x_0 t}\sin\left(y_0(t^2-1)^{1/2}\right)\frac{\mathrm{d}t}{t},\quad x_0\geqslant 0,\tag{5.4a}$$

$$\frac{I}{Q}=\mathrm{e}^{-y_0}-\frac{1}{\pi}\int_1^{\infty}\mathrm{e}^{x_0 t}\sin\left(y_0(t^2-1)^{1/2}\right)\frac{\mathrm{d}t}{t},\quad x_0\leqslant 0.\tag{5.4b}$$

By differentiation of (5.4a, b) with respect to x_0, we obtain

$$\frac{\partial}{\partial x_0}\left(\frac{I}{Q}\right) = -\frac{1}{\pi}\int_1^\infty \mathrm{e}^{-|x_0|t}\sin\left(y_0(t^2-1)^{1/2}\right)\mathrm{d}t$$

$$= -\frac{1}{\pi}\int_0^\infty \frac{t}{(t^2+1)^{1/2}}\exp\left[-|x_0|(t^2+1)^{1/2}\right]\sin(y_0 t)\mathrm{d}t$$

$$= -\frac{y_0}{\pi}\frac{K_1\left((x_0^2+y_0^2)^{1/2}\right)}{(x_0^2+y_0^2)^{1/2}} \tag{5.5}$$

where the latter Fourier sine transform has been taken from [19, form. 2.4(36)]. Both results (5.4a) and (5.4b) can now be reduced to (5.2) in an obvious manner.

Next we consider the limiting case $\lambda = 0$, corresponding to a surface recombination velocity $v = 0$, for which the solution of the diffusion problem (2.4) has been lacking so far. When $\lambda = 0$, the factorization (3.12) becomes trivial, yielding $H_\pm(w) = 1$. As a consequence it is found from (3.1) and (3.22) that the solution for the concentration $p(x, y)$ is given by

$$p(x,y) = \frac{1}{2\pi}K_0\left(\left((x-x_0)^2+(y-y_0)^2\right)^{1/2}\right) - \frac{1}{2\pi}K_0\left(\left((x-x_0)^2+(y+y_0)^2\right)^{1/2}\right)$$

$$+\frac{1}{4\pi^2 \mathrm{i}}\int_{-\infty+\mathrm{i}c}^{\infty+\mathrm{i}c}\mathrm{d}w$$

$$\times\int_{-\infty+\mathrm{i}b}^{\infty+\mathrm{i}b}\frac{\exp\left[-\mathrm{i}xw+\mathrm{i}x_0 z-y(w^2+1)^{1/2}-y_0(z^2+1)^{1/2}\right]}{(1-\mathrm{i}w)^{1/2}(1+\mathrm{i}z)^{1/2}}\frac{\mathrm{d}z}{z-w},$$

$$-1<b<c<1. \tag{5.6}$$

The latter formidable double integral is very similar to the one encountered by Clemmow [22] in his treatment of the diffraction of a cylindrical wave by a perfectly conducting half-plane. By Clemmow's analysis, properly modified, the double integral in (5.6) can be reduced to a sum of two single integrals of error-function type. Omitting the details we present the final result for $p(x, y)$:

$$p(x,y) = \frac{\mathrm{e}^{-R_1}}{\pi}\int_0^{\epsilon_1}\frac{\mathrm{e}^{-t^2}}{(t^2+2R_1)^{1/2}}\mathrm{d}t + \frac{\mathrm{e}^{-R_2}}{\pi}\int_0^{\epsilon_2}\frac{\mathrm{e}^{-t^2}}{(t^2+2R_2)^{1/2}}\mathrm{d}t, \tag{5.7}$$

in which

$$R_1 = \left\{(x-x_0)^2+(y-y_0)^2\right\}^{1/2},\quad R_2 = \left\{(x-x_0)^2+(y+y_0)^2\right\}^{1/2},$$
$$\epsilon_1 = (r+r_0-R_1)^{1/2},\quad \epsilon_2 = (r+r_0-R_2)^{1/2}\operatorname{sgn}\left[\cos\tfrac{1}{2}(\theta+\theta_0)\right]; \tag{5.8}$$

here, (r, θ) and (r_0, θ_0) denote polar coordinates defined by $x = r\cos\theta$, $y = r\sin\theta$ and $x_0 = r_0\cos\theta_0$, $y_0 = r_0\sin\theta_0$, with $0 \leqslant \theta,\ \theta_0 \leqslant \pi$. Alternatively and more simply, the solution (5.7) may be obtained directly from the final result in [22, Eqn. (30)] by exploiting the relationship between our diffusion problem (2.4) with $\lambda = 0$ and Clemmow's diffraction problem.

We now come to the evaluation of the current I/Q in the case $\lambda = 0$. As a first approach one might substitute (5.7) into (2.5); however, the further reduction of the current integral gets quite involved, although it does lead to the results (5.11a, b) below. Therefore we prefer to start from the expressions (4.5) and (4.7), taking the limit as $\lambda \to 0$. Since $F(0, t) = 0$ by (4.6), it is immediately seen that in the limiting case $\lambda = 0$ the current is given by

$$\frac{I}{Q} = \frac{1}{\pi}\int_1^\infty \frac{e^{-x_0 t}\cos\left(y_0(t^2-1)^{1/2}\right)}{t(t-1)^{1/2}}\,dt, \quad x_0 \geqslant 0, \tag{5.9a}$$

$$\frac{I}{Q} = e^{-y_0} - \frac{1}{\pi}\int_1^\infty \frac{e^{x_0 t}\sin\left(y_0(t^2-1)^{1/2}\right)}{t(t+1)^{1/2}}\,dt, \quad x_0 \leqslant 0. \tag{5.9b}$$

In the Appendix it is shown that the integrals (5.9a, b) can be expressed in terms of the complementary error function defined by

$$\operatorname{erfc}(z) = \frac{2}{\pi^{1/2}}\int_z^\infty e^{-t^2}dt. \tag{5.10}$$

Thus we have from (A.9), (A.10),

$$\frac{I}{Q} = \tfrac{1}{2}\,e^{-y_0}\operatorname{erfc}\left[(r_0 - y_0)^{1/2}\right] + \tfrac{1}{2}\,e^{y_0}\operatorname{erfc}\left[(r_0 + y_0)^{1/2}\right], \quad x_0 \geqslant 0, \tag{5.11a}$$

$$\frac{I}{Q} = e^{-y_0} - \tfrac{1}{2}\,e^{-y_0}\operatorname{erfc}\left[(r_0 - y_0)^{1/2}\right] + \tfrac{1}{2}\,e^{y_0}\operatorname{erfc}\left[(r_0 + y_0)^{1/2}\right]$$

$$= \tfrac{1}{2}\,e^{-y_0}\operatorname{erfc}\left[-(r_0 - y_0)^{1/2}\right] + \tfrac{1}{2}\,e^{y_0}\operatorname{erfc}\left[(r_0 + y_0)^{1/2}\right], \quad x_0 \leqslant 0, \tag{5.11b}$$

in which $r_0 = (x_0^2 + y_0^2)^{1/2}$. This completes the discussion of the limiting case $\lambda = 0$.

6. Asymptotics of the current I/Q as $x_0 \to \infty$

Although suitable for numerical calculations, the expressions (4.5) and (4.7) for the current I/Q are too complicated to provide insight into the behaviour of the current as a function of the parameters x_0, y_0, λ. Therefore it is desirable to establish some simple approximate solution for the current, which is valid in a parameter range of practical interest. In this section we derive the asymptotic expansion of the current I/Q as $x_0 \to \infty$. It is expected that the asymptotic approximation will be usable if the injection point is more than a few diffusion lengths away from the edge of the Schottky diode.

We first consider the case $\lambda > 0$. Then the integral (4.5) can be shortly expressed as

$$\frac{I}{Q} = \int_1^{\infty} (t-1)^{1/2} G(t)\, \mathrm{e}^{-x_0 t} \mathrm{d}t, \tag{6.1}$$

in which the function $G(t)$ is regular in the vicinity of $t = 1$. The asymptotic expansion of the latter integral is derived by the method of Laplace. The basic idea in this method is that, if $x_0 \gg 1$, the main contribution to the integral (6.1) stems from the immediate vicinity of the lower limit $t = 1$. The asymptotic expansion is obtained now by replacing $G(t)$ by its Taylor expansion around $t = 1$, followed by a term-by-term integration. Retaining only two terms of the Taylor expansion, we find

$$\frac{I}{Q} = \tfrac{1}{2}\pi^{1/2} G(1) \frac{\mathrm{e}^{-x_0}}{x_0^{3/2}} \left[1 + \frac{3}{2} \frac{G'(1)}{G(1)} x_0^{-1} + O\left(x_0^{-2}\right) \right], \quad x_0 \to \infty. \tag{6.2}$$

Next, by inserting the actual values of $G(1)$ and $G'(1)$ implied by (4.5), we have the asymptotic approximation to second order

$$\frac{I}{Q} = f(\lambda) \frac{(1+\lambda)(1+\lambda y_0)}{\lambda^2} \frac{\mathrm{e}^{-x_0}}{\pi^{1/2} x_0^{3/2}}$$

$$\times \left[1 - \tfrac{3}{2} \left\{ \tfrac{1}{2} + \frac{2}{\lambda^2} + g(\lambda) + y_0^2 \frac{1 + \lambda y_0/3}{1 + \lambda y_0} \right\} x_0^{-1} + O\left(x_0^{-2}\right) \right], \quad x_0 \to \infty. \tag{6.3}$$

Here the functions $f(\lambda)$ and $g(\lambda)$ are defined by

$$f(\lambda) = (1+\lambda)^{-1/2} \exp[F(\lambda, 1)] = (1+\lambda)^{-1/2} \exp\left[\frac{1}{\pi} \int_0^{\pi/2} \log(1 + \lambda \sin \varphi) \mathrm{d}\varphi \right], \tag{6.4}$$

$$g(\lambda) = -\frac{\partial F}{\partial t}(\lambda, 1) = \frac{\lambda}{\pi} \int_0^{\pi/2} \frac{\sin \varphi \cos^2 \varphi}{1 + \lambda \sin \varphi} \mathrm{d}\varphi, \tag{6.5}$$

in which the right terms have been obtained by means of (4.6). It is easily seen that both functions $f(\lambda)$ and $g(\lambda)$ are monotonic for $\lambda \geqslant 0$: $f(\lambda)$ decreases from $f(0) = 1$ to $f(\infty) = 2^{-1/2}$, while $g(\lambda)$ increases from $g(0) = 0$ through $g(1) = \pi^{-1} - 1/4$ to $g(\infty) = 1/4$. The elementary integral (6.5) can be evaluated, yielding

$$g(\lambda) = \begin{cases} \dfrac{1}{\lambda^2} \left[\tfrac{1}{4}\lambda^2 - \tfrac{1}{2} + \dfrac{1}{\pi} \left\{ \lambda + (1-\lambda^2)^{1/2} \arccos \lambda \right\} \right], & 0 \leqslant \lambda \leqslant 1, \\ \dfrac{1}{\lambda^2} \left[\tfrac{1}{4}\lambda^2 - \tfrac{1}{2} + \dfrac{1}{\pi} \left\{ \lambda - (\lambda^2-1)^{1/2} \log\left(\lambda + (\lambda^2-1)^{1/2}\right) \right\} \right], & \lambda \geqslant 1. \end{cases} \tag{6.6}$$

As a side-remark, the integral occurring in (6.4) can be expressed in terms of Clausen functions by means of Lewin [23, p. 308, form. (36), (38)], viz.

$$h(\lambda)=\frac{1}{\pi}\int_0^{\pi/2}\log(1+\lambda\sin\varphi)\mathrm{d}\varphi$$

$$=\tfrac{1}{2}\log\left(\tfrac{1}{2}+\tfrac{1}{2}(1-\lambda^2)^{1/2}\right)+\frac{1}{2\pi}\arcsin\lambda\log\left(\frac{1-(1-\lambda^2)^{1/2}}{1+(1-\lambda^2)^{1/2}}\right)$$

$$+\frac{1}{\pi}\mathrm{Cl}_2(\arcsin\lambda)+\frac{1}{\pi}\mathrm{Cl}_2(\pi-\arcsin\lambda),\quad 0\leqslant\lambda\leqslant 1, \tag{6.7a}$$

$$h(\lambda)=\tfrac{1}{2}\log\left(\frac{\lambda}{2}\right)+\frac{1}{\pi}\mathrm{Cl}_2(\arcsin\lambda^{-1})+\frac{1}{\pi}\mathrm{Cl}_2(\pi-\arcsin\lambda^{-1}),\quad \lambda\geqslant 1, \tag{6.7b}$$

in which the Clausen function Cl_2 is defined by [23, p. 102]

$$\mathrm{Cl}_2(\varphi)=\sum_{n=1}^{\infty}\frac{\sin(n\varphi)}{n^2}. \tag{6.8}$$

It is interesting to note that also the integral occurring in (3.14) can be expressed in terms of (complex) Clausen functions by means of [23, p. 306, form. (13)]. From $\mathrm{Cl}_2(\pi/2)=G$, Catalan's constant, we have the special value $f(1)=\frac{1}{2}\exp[2G/\pi]=0.89581$ to five decimal places.

It is pointed out that the approximation (6.3) fails when λ gets close to 0, and breaks down completely in the limiting case $\lambda=0$. It is easily verified that for small values of λ, say $0<\lambda\leqslant 1$, the error term $O(x_0^{-2})$ in (6.3) is to be replaced by $O(\lambda^{-4}x_0^{-2})$. Hence, the approximation applies only under the stronger restriction $\lambda^2 x_0 \gg 1$. Thus it is obvious that the approximation (6.3) is not uniformly valid in λ over the range $\lambda>0$. In the limiting case $\lambda=0$ the asymptotics of the current I/Q can be readily determined from the solution (5.11a). By use of the well-known asymptotic expansion

$$\mathrm{erfc}(z)=\frac{\mathrm{e}^{-z^2}}{\pi^{1/2}z}\left[1-\tfrac{1}{2}z^{-2}+O(z^{-4})\right],\quad z\to\infty, \tag{6.9}$$

we are led to the asymptotic approximation to second order

$$\frac{I}{Q}=\frac{\mathrm{e}^{-x_0}}{\pi^{1/2}x_0^{1/2}}\left[1-\tfrac{1}{2}(1+y_0^2)x_0^{-1}+O(x_0^{-2})\right],\quad x_0\to\infty. \tag{6.10}$$

A comparison of (6.3) and (6.10) shows again the non-uniformity in λ: for $\lambda>0$ the current decays with distance x_0 according to $x_0^{-3/2}\,\mathrm{e}^{-x_0}$, whereas for $\lambda=0$ the current decay is governed by $x_0^{-1/2}\,\mathrm{e}^{-x_0}$. This different behaviour was first noticed by Davidson and Dimitriadis [9] for the limiting cases $\lambda=\infty$ and $\lambda=0$. However, these authors did not show explicitly how their correct result for $\lambda=0$ was obtained.

In the limiting case $\lambda=\infty$ the current I/Q is given by (5.2). Through repeated integration by parts in (5.2), using the well-known recurrence relation for the modified

Bessel functions K_m, we deduce the complete asymptotic expansion

$$\frac{I}{Q} \sim \frac{y_0}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{(2m)!}{2^m m!} \frac{\left(x_0^2 + y_0^2\right)^{m/2}}{x_0^{2m+1}} K_m\left(\left(x_0^2 + y_0^2\right)^{1/2}\right), \quad x_0 \to \infty. \tag{6.11}$$

Here the Bessel functions K_m may be replaced by their asymptotic expansions, thus leading to the asymptotic approximation to second order

$$\frac{I}{Q} = 2^{-1/2} y_0 \frac{e^{-x_0}}{\pi^{1/2} x_0^{3/2}} \left[1 - \left(\frac{9}{8} + \tfrac{1}{2} y_0^2\right) x_0^{-1} + O\left(x_0^{-2}\right)\right], \quad x_0 \to \infty. \tag{6.12}$$

This result is identical to the asymptotic expansion derived by Ioannou and Dimitriadis [11]. Notice that (6.12) also follows by setting $\lambda = \infty$ in (6.3).

Finally, we present a simple uniform approximation for the current that is uniformly valid in λ over the entire range $\lambda \geqslant 0$. To that end we start from the integral (6.1) in which the function $G(t)$, obtainable from (4.5), is replaced by

$$\frac{2}{\pi}(1+\lambda)^{1/2} \exp[F(\lambda, 1)] \frac{1 + \lambda y_0}{\lambda^2 + 2(t-1)} = \frac{2}{\pi} f(\lambda) \frac{(1+\lambda)(1+\lambda y_0)}{\lambda^2 + 2(t-1)}. \tag{6.13}$$

This amounts to a replacement of $G(t)$ by $G(1)$, except for the denominator $\lambda^2 + t^2 - 1$ in (4.5) which is replaced by $\lambda^2 + 2(t-1)$. Then, in the spirit of Laplace's method, we obtain the following approximation:

$$\frac{I}{Q} \approx \frac{2}{\pi} f(\lambda)(1+\lambda)(1+\lambda y_0) \int_1^\infty \frac{(t-1)^{1/2} e^{-x_0 t}}{\lambda^2 + 2(t-1)} dt, \quad x_0 \to \infty. \tag{6.14}$$

The latter integral is rewritten as

$$\int_1^\infty (t-1)^{1/2} e^{-x_0 t} dt \int_0^\infty \exp\left[-\left(\lambda^2 + 2(t-1)\right)s\right] ds$$

$$= e^{-x_0} \int_0^\infty e^{-\lambda^2 s} ds \int_0^\infty t^{1/2} \exp\left[-(x_0 + 2s)t\right] dt$$

$$= \tfrac{1}{2}\pi^{1/2} e^{-x_0} \int_0^\infty \frac{e^{-\lambda^2 s}}{(x_0 + 2s)^{3/2}} ds$$

which, via an integration by parts, can be expressed in terms of a complementary error function. In this manner we obtain the uniform approximation to first order

$$\frac{I}{Q} \approx f(\lambda)(1+\lambda)(1+\lambda y_0) \frac{e^{-x_0}}{\pi^{1/2} x_0^{1/2}}$$

$$\times \left\{1 - \pi^{1/2} \lambda (x_0/2)^{1/2} e^{\lambda^2 x_0/2} \operatorname{erfc}\left[\lambda (x_0/2)^{1/2}\right]\right\}, \quad x_0 \to \infty. \tag{6.15}$$

If $\lambda = 0$, the approximation (6.15) reduces to the leading term of the expansion (6.10). If

$\lambda > 0$ and $\lambda^2 x_0 \gg 1$, the erfc function may be replaced by its asymptotic expansion (6.9), whereupon the approximation (6.15) reduces to the leading term of the expansion (6.3). This explains why the approximation (6.15) is called uniform to first order.

The numerical accuracy of the approximations of this section is investigated in the companion paper [18], through a comparison of numerical results based on the approximations and on the exact solution of Section 4.

Appendix: Evaluation of the integrals (5.9)

This Appendix deals with the evaluation of the integrals

$$I_1 = \frac{1}{\pi}\int_1^\infty \frac{e^{-x_0 t}\cos\left(y_0(t^2-1)^{1/2}\right)}{t(t-1)^{1/2}}\,dt, \tag{A.1}$$

$$I_2 = \frac{1}{\pi}\int_1^\infty \frac{e^{-x_0 t}\sin\left(y_0(t^2-1)^{1/2}\right)}{t(t+1)^{1/2}}\,dt, \tag{A.2}$$

in which $x_0 \geqslant 0$, $y_0 \geqslant 0$. Introduce polar coordinates (r_0, θ_0) defined by $x_0 = r_0 \cos\theta_0$, $y_0 = r_0 \sin\theta_0$, with $0 \leqslant \theta_0 \leqslant \pi/2$, and apply the substitution $t = \cosh s$, then the integrals I_1, I_2 can be expressed as

$$I_1 = \frac{1}{2^{1/2}\pi}\int_{-\infty}^\infty \exp[-r_0\cosh(s - i\theta_0)]\frac{\cosh\frac{1}{2}s}{\cosh s}\,ds, \tag{A.3}$$

$$I_2 = \frac{1}{2^{1/2}\pi i}\int_{-\infty}^\infty \exp[-r_0\cosh(s - i\theta_0)]\frac{\sinh\frac{1}{2}s}{\cosh s}\,ds. \tag{A.4}$$

We now consider the combinations

$$I_1 \pm I_2 = \frac{1}{2^{1/2}\pi}\int_{-\infty}^\infty \frac{\exp[-r_0\cosh(s - i\theta_0)]}{\cosh\frac{1}{2}s \pm i\sinh\frac{1}{2}s}\,ds, \tag{A.5}$$

in which we replace the variable s by $s + i\theta_0$. The resulting integral is rewritten as

$$\begin{aligned} I_1 \pm I_2 &= \frac{1}{2^{1/2}\pi}\int_0^\infty \exp[-r_0\cosh s] \\ &\quad\cdot\left\{\left[\cosh\tfrac{1}{2}(s+i\theta_0) \pm i\sinh\tfrac{1}{2}(s+i\theta_0)\right]^{-1}\right. \\ &\quad\left.+\left[\cosh\tfrac{1}{2}(s-i\theta_0) \mp i\sinh\tfrac{1}{2}(s-i\theta_0)\right]^{-1}\right\}ds \\ &= \frac{2^{1/2}}{\pi}\left(\cos\tfrac{1}{2}\theta_0 \mp \sin\tfrac{1}{2}\theta_0\right)\int_0^\infty \exp[-r_0\cosh s]\frac{\cosh\frac{1}{2}s}{\cosh s \mp \sin\theta_0}\,ds. \end{aligned} \tag{A.6}$$

Through the substitution $2^{1/2}\sinh\frac{1}{2}s=\sigma$, the latter integral transforms into

$$I_1\pm I_2=\frac{2}{\pi}\left(\cos\tfrac{1}{2}\theta_0\mp\sin\tfrac{1}{2}\theta_0\right)\mathrm{e}^{-r_0}\int_0^\infty\frac{\exp\left[-r_0\sigma^2\right]}{\sigma^2+1\mp\sin\theta_0}\,\mathrm{d}\sigma. \tag{A.7}$$

Finally, by setting

$$\frac{1}{\sigma^2+1\mp\sin\theta_0}=\int_0^\infty\exp\left[-\left(\sigma^2+1\mp\sin\theta_0\right)t\right]\mathrm{d}t$$

in (A.7), and by interchanging the order of integration, we find

$$\begin{aligned}I_1\pm I_2&=\pi^{-1/2}\left(\cos\tfrac{1}{2}\theta_0\mp\sin\tfrac{1}{2}\theta_0\right)\mathrm{e}^{-r_0}\int_0^\infty\frac{\exp\left[-(1\mp\sin\theta_0)t\right]}{\left(r_0+t\right)^{1/2}}\,\mathrm{d}t\\&=\frac{\cos\tfrac{1}{2}\theta_0\mp\sin\tfrac{1}{2}\theta_0}{\left(1\mp\sin\theta_0\right)^{1/2}}\exp\left[\mp r_0\sin\theta_0\right]\frac{2}{\pi^{1/2}}\int_{r_0^{1/2}(1\mp\sin\theta_0)^{1/2}}^\infty\mathrm{e}^{-t^2}\mathrm{d}t\\&=\mathrm{e}^{\mp y_0}\,\mathrm{erfc}\left[\left(r_0\mp y_0\right)^{1/2}\right].\end{aligned} \tag{A.8}$$

Hence we have

$$I_1=\tfrac{1}{2}\mathrm{e}^{-y_0}\,\mathrm{erfc}\left[\left(r_0-y_0\right)^{1/2}\right]+\tfrac{1}{2}\mathrm{e}^{y_0}\,\mathrm{erfc}\left[\left(r_0+y_0\right)^{1/2}\right], \tag{A.9}$$

$$I_2=\tfrac{1}{2}\mathrm{e}^{-y_0}\,\mathrm{erfc}\left[\left(r_0-y_0\right)^{1/2}\right]-\tfrac{1}{2}\mathrm{e}^{y_0}\,\mathrm{erfc}\left[\left(r_0+y_0\right)^{1/2}\right], \tag{A.10}$$

where $r_0=(x_0^2+y_0^2)^{1/2}$.

References

[1] R.A. Smith, *Semiconductors*, Cambridge University Press, Cambridge, 1959.

[2] W. van Roosbroeck, Injected current carrier transport in a semi-infinite semiconductor and the determination of lifetimes and surface recombination velocities, *J. Appl. Phys.* 26 (1955) 380–391.

[3] W.H. Hackett, Electron-beam excited minority-carrier diffusion profiles in semiconductors, *J. Appl. Phys.* 43 (1972) 1649–1654.

[4] F. Berz and H.K. Kuiken, Theory of life time measurements with the scanning electron microscope: steady state, *Solid-State Electron.* 19 (1976) 437–445.

[5] H.K. Kuiken, Theory of lifetime measurements with the scanning electron microscope: transient analysis, *Solid-State Electron.* 19 (1976) 447–450.

[6] C. van Opdorp, Methods of evaluating diffusion lengths and near-junction luminescence-efficiency profiles from SEM scans, *Philips Res. Repts.* 32 (1977) 192–249.

[7] C. Donolato, On the analysis of diffusion length measurements by SEM, *Solid-State Electron.* 25 (1982) 1077–1081.

[8] H.J. Leamy, Charge collection scanning electron microscopy, *J. Appl. Phys.* 53 (1982) R51–R80.

[9] S.M. Davidson and C.A. Dimitriadis, Advances in the electrical assessment of semiconductors using the scanning electron microscope, *J. Microscopy* 118 (1980) 275–290.

[10] D.E. Ioannou, A SEM-EBIC minority-carrier lifetime-measurement technique, *J. Phys. D: Appl. Phys.* 13 (1980) 611–616.

[11] D.E. Ioannou and C.A. Dimitriadis, A SEM-EBIC minority-carrier diffusionlength measurement technique, *IEEE Trans. Electron. Devices* 29 (1982) 445–450.
[12] A.E. Heins, The scope and limitations of the method of Wiener and Hopf, *Comm. Pure Appl. Math.* 9 (1956) 447–466.
[13] J. Bazer and S.N. Karp, Propagation of plane electromagnetic waves past a shoreline, *J. Res. Nat. Bur. Stand.* 66 D (1962) 319–334.
[14] T.B.A. Senior, Diffraction by a semi-infinite metallic sheet, *Proc. Roy. Soc. (London)* A 213 (1952) 436–458.
[15] T.B.A. Senior, Half plane edge diffraction, *Radio Sci.* 10 (1975) 645–650.
[16] T.B.A. Senior, Diffraction tensors for imperfectly conducting edges, *Radio Sci.* 10 (1975) 911–919.
[17] B. Noble, *Methods based on the Wiener-Hopf technique*, Pergamon Press, London, 1958.
[18] H.K. Kuiken and C. van Opdorp, Evaluation of diffusion length and surface-recombination velocity from a planar-collector geometry EBIC scan, to appear in *J. Appl. Phys.*
[19] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of integral transforms*, Vol. I, McGraw-Hill, New York, 1954.
[20] G.D. Maliuzhinets, Excitation, reflection and emission of surface waves from a wedge with given face impedances, *Sov. Phys. Dokl.* 3 (1958) 752–755.
[21] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products*, Academic Press, New York, 1965.
[22] P.C. Clemmow, A note on the diffraction of a cylindrical wave by a perfectly conducting half-plane, *Quart. J. Mech. Appl. Math.* 3 (1950) 377–384.
[23] L. Lewin, *Polylogarithms and associated functions*, Elsevier North Holland, New York, 1981.